Well-Behaved Link Delay Functions

David Bernstein
Massachusetts Institute of Technology

Tony E.Smith
University of Pennsylvania

November 8, 1993

presented at
The North American Meetings of the RSAI
DRAFT FOR DISCUSSANTS ONLY

Abstract

This paper develops a class of link delay functions that can be used in a variety of different types of dynamic traffic assignment models. We show that these link delay functions are well-behaved in the sense that their corresponding exit time functions are strictly increasing. Hence, these link delay functions prevent "overtaking" or violations of "first-in first-out". In addition, we explore the properties of this class of functions by considering some small numerical examples.

1 Introduction

Dynamic traffic assignment models take two different forms. In the first, the intent is to model the *default* path and departure-time choices of drivers. These models are generally referred to as *simultaneous route and departure-time choice equilibrium* models or *dynamic user equilibrium* models [see, for example, the recent work by Smith and Ghali (1990), Drissi-Kaitouni (1990), Cascetta (1991), Janson (1991), Friesz et al. (1993), Ran et al. (1993), and

Bernstein et al. (1993)]. In the second, the intent is to model the *adjustments* that drivers make to their path and departure-time in response to changing conditions. These models are generally referred to simply as *dynamic traffic assignment* models (for IVHS) [see, for example, the recent work by Chang and Mahmassani (1988) and Ben-Akiva et al. (1991)].

Though these models can be (and usually are) quite different in nature, both require some form of link delay/performance function.[1] Unfortunately, however, many existing link delay/performance functions are inappropriate since they are describe the average performance of a link over time. This includes many of the delay/performance functions that have been used in static traffic assignment models [see, for example, Branston (1976)] as well as those that have been used in more traditional traffic flow analyses [see, for example Mosher (1963), Kraft and Wohl (1967), Boardman and Lave (1977)].

Not surprisingly, this led many researchers to develop "new" link delay/performance functions for their dynamic traffic assignment models. Unfortunately, this task proved to be more dificult than was originally suspected. In particular, various researchers quickly realized that not all link delay/performance functions prevent *overtaking* (i.e., enforce a first-in first-out discipline). In general, this has led to three different responses: ignore overtaking altogether, include constraints (of some form) which prohibit overtaking, or use well-behaved link delay functions which eliminate the possibility of overtaking.

This paper describes a class of link delay functions which are well-behaved in the sense defined above. We begin by defining these link-delay functions and proving that their corresponding exit time functions are strictly increasing (as desired). We then consider several small examples to illustrate the properties of these delay functions.

2 Regular Delay Operators

We consider a single arc with a known (integrable) entry flow rate at time t which is denoted by $h(t)$. Further, we let $x(t)$ denote the number of vehicles on the arc at time t, and $D(t)$ denote the delay (i.e., travel time) experienced

[1]The one exception is models that are based on a microsimulation of vehicle following and lane-changing behavior.

by vehicles entering the arc at time t. Then, if we let $T(t)$ denote the set of (Lebesgue measurable) departure times of all commuters still on a at time t then we know that

$$x(t) = \int_{T(t)} h(s)ds \tag{1}$$

and that the exit time of vehicles that enter the link at time t is given by $\tau(t) = t + D(t)$.

We now introduce the following:

Definition 2.1 *An arc delay function, D, is said to be* regular *iff there exists a constant, $\beta > 0$, and a continuous delay-impact function, $\phi : R_+ \to R_+$, with $\phi(0) = 0$, such that for all $t \geq 0$,*

$$D(t) = \int_{T(t)} \phi[x(s)]h(s)ds + \beta \tag{2}$$

Thus, an arc delay function is regular if the delay at time t is the sum of an *uncongested delay time*, β and a *congested delay time* composed of the cumulative effects of all commuters currently on link a. The specific delay impact of each commuter entering the link at a prior time, $s \in T(t)$, is assumed to be a function, $\phi[x(s)]$, of the number of commuters already on the link at time s. If ϕ is constant then each commuter has the same delay impact on subsequent commuters. If ϕ is increasing then commuters entering under more congested conditions have a greater delay effect on subsequent commuters.

Before moving to the main result we first develop the following:

Lemma 2.1 *For any differentiable, invertible function, $f : R \to R$, with derivative, $f\prime$:*

$$f^{-1}[f(t)] \equiv t \Rightarrow [f^{-1}]'(z) \equiv 1/f'[f^{-1}(z)] \tag{3}$$

Proof. Simply observe that:

$$\begin{aligned} f^{-1}[f(t)] \equiv t \quad &\Rightarrow \quad [f^{-1}]'[f(t)] \cdot f'(t) \equiv 1 \\ &\Rightarrow \quad [f^{-1}]'(z) \cdot f'[f^{-1}(z)] \equiv 1, \quad z = f(t) \\ &\Rightarrow \quad [f^{-1}]'(z) \equiv 1/f'[f^{-1}(z)]. \end{aligned} \tag{4}$$

Q.E.D.

With this result, we now demonstrate the following:

Theorem 2.1 *For any regular arc delay function, D, the resulting arc exit time function, τ, is strictly increasing (and hence overtaking does not occur).*

Proof. Let $t_1 = \tau(0)$ denote the time of the first exit from a. Then by assumption $x(0) = 0 \Rightarrow T(0) = \emptyset$ so that by (2) $t_1 = D(0) = \beta$. Moreover, it also follows that

$$t \in [0, t_1] \Rightarrow T(t) = [0, t] \tag{5}$$

so that by (1),

$$x(t) = \int_0^t h(s)ds, \quad tin[0, t_1]. \tag{6}$$

This in turn implies from (2) that for each $t \in [0, t_1]$, the exit time, $\tau_1(t)[= \tau(t)]$, is given by

$$\begin{aligned} \tau_1(t) &= t + D(t) \qquad (7) \\ &= t + \int_0^t \phi[x(s)]h(s)ds + \beta \\ &= t + \int_0^t \phi\left[\int_0^s h(z)dz\right] h(s)ds + \beta. \end{aligned}$$

Hence, the continuity of h and ϕ imply that τ_1 is differentiable with derivative, τ_1', given for all $t \in [0, t_1]$ by

$$\tau_1'(t) = 1 + \phi\left[\int_0^t h(z)dz\right] h(t) > 0 \tag{8}$$

which implies that the exit time function, $\tau_1(= \tau)$, is strictly increasing on $[0, t_1]$.

This means that on the interval $[t_1, t_2] \equiv [\tau_1(0), \tau_1(t_1)]$ there must exist a well-defined inverse function $\tau_1^{-1} : [t_1, t_2] \to [0, t_1]$. Thus, for any time $t \in [t_1, t_2]$, the travelers on arc a at time t are precisely those who departed during the interval $[\tau_1^{-1}(t), t]$. That is,

$$t \in [t_1, t_2] \Rightarrow T(t) = [\tau_1^{-1}(t), t]. \tag{9}$$

Hence, if the exit time for departures during the interval $[t_1, t_2]$ is denoted by τ_2, then for all $t \in [t_1, t_2]$

$$\begin{aligned} \tau_2(t) &= t + \int_{\tau_1^{-1}(t)}^{t_1} \phi[x(s)]h(s)ds + \beta \\ &= t + \int_{\tau_1^{-1}(t)}^{t_1} \phi\left[\int_{T(s)} h(z)dz\right] h(s)ds + \beta. \end{aligned} \tag{10}$$

Next obsevre that by definition, $t \in [t_1, t_2] \Rightarrow \tau_1^{-1}(t) \leq t_1 \leq t$. But, if $s \in [\tau_1^{-1}(t), t_1]$ then $T(s) = [0, s]$ by (5). In addition, if $s \in [t_1, t]$ then $T(s) = [\tau_1^{-1}(s), s]$ by (9), so that by breaking the outer integral in (11) into two parts we obtain

$$\begin{aligned} \tau_2(t) &= t + \int_{\tau_1^{-1}(t)}^{t_1} \phi[x(s)]h(s)ds + \int_{t_1}^{t} \phi[x(s)]h(s)ds + \beta \\ &= t + \int_{\tau_1^{-1}(t)}^{t_1} \phi\left[\int_0^s h(z)dz\right] h(s)ds + \int_{t_1}^{t} \phi\left[\int_{\tau_1^{-1}(s)}^{s} h(z)dz\right] h(s)ds + \beta. \end{aligned} \tag{11}$$

Note also that $\tau_1^{-1}(t_1) = 0$, which, together with (8) and (12) implies that

$$\begin{aligned} \tau_2(t) &= t_1 + \int_0^{t_1} \phi\left[\int_0^s h(z)dz\right] h(s)ds + 0 + \beta \\ &= \tau_1(t_1). \end{aligned} \tag{12}$$

Hence, by differentiating (12) and applying (8) and (3) we obtain

$$\begin{aligned} \tau_2'(t) &= 1 - \phi\left[\int_0^{\tau_1^{-1}(t)} h(z)dz\right] h[\tau_1^{-1}(t)] + \left(\tau_1'[\tau_1^{-1}(t)]\right)^{-1} + \phi\left[\int_{\tau_1^{-1}(t)}^{t} h(z)dz\right] h(t) \\ &= \left(1 - \frac{\phi\left[\int_0^{\tau_1^{-1}(t)} h(z)dz\right] h[\tau_1^{-1}(t)]}{1 + \phi\left[\int_0^{\tau_1^{-1}(t)} h(z)dz\right] h[\tau_1^{-1}(t)]}\right) + \phi\left[\int_{\tau_1^{-1}(t)}^{t} h(z)dz\right] h(t) \end{aligned} \tag{13}$$

and it follows from the nonegativity of ϕ and h that the first term is positive. Hence, we may conclude that

$$\tau_2'(t) > \phi\left[\int_{\tau_1^{-1}(t)}^{t} h(z)dz\right] h(t) \geq 0 \tag{14}$$

for all $t \in [t_1, t_2]$. Thus, τ_2 is also strictly increasing which again implies that the function $\tau_2^{-1} : [\tau_2(t_1), \tau_2(t_2)] \longrightarrow [t_1, t_2]$, is well defined.

We now proceed by induction as follows. Choose any $n > 2$ and hypothesize [as an extension of (12), (13), and (14)] that for each $k = 2, \ldots, n$ there exist invertible functions, $\tau_k : [t_{k-1}, t_k] \longrightarrow [\tau_k(t_{k-1}), \tau_k(t_k)] = [t_k, t_{k+1}]$ satisfying the following three conditions for all $t \in [t_{k-1}, tk]$ and $k = 2, \ldots, n$ [where we set $\tau_0^{-1} \equiv 0$ when $k = 2$],

$$\begin{aligned}
\tau_k(t) &= t + \int_{\tau_{k-1}^{-1}(t)}^{t} \phi\left[\int_{T(s)} h(z)dz\right] + \beta \qquad (15)\\
&= t + \int_{\tau_{k-1}^{-1}(t)}^{t_{k-1}} \phi\left[\int_{\tau_{k-2}^{-1}(s)}^{s} h(z)dz\right] h(s)ds + \int_{t_{k-1}}^{t} \phi\left[\int_{\tau_{k-1}^{-1}(s)}^{s} h(z)dz\right] h(s)ds + \beta
\end{aligned}$$

$$\tau_k'(t) > \phi\left[\int_{\tau_{k-1}^{-1}(t)}^{t} h(z)dz\right] h(t) \geq 0 \tag{16}$$

$$\tau_{k-1}(t_k) = \tau_k(t_k). \tag{17}$$

Under these hypotheses, we wish to show that the function $\tau_{n+1} : [t_n, t_{n+1}] \longrightarrow [\tau_{n+1}(t_n), \tau_{n+1}(t_{n+1})]$, defined for all $t \in [t_n, t_{n+1}]$ by

$$\tau_{n+1}(t) = t + \int_{\tau_n^{-1}(t)}^{t_n} \phi\left[\int_{\tau_{n-1}^{-1}(s)}^{s} h(z)dz\right] h(s)ds + \int_{t_n}^{t} \phi\left[\int_{\tau_n^{-1}(s)}^{s} h(z)dz\right] h(s)ds + \beta \tag{18}$$

also satisfies the following conditions paralleling (16) and (17),

$$\tau_{n+1}'(t) > \phi\left[\int_{\tau_n^{-1}(t)}^{t} h(z)dz\right] h(t) \geq 0 \quad , t \in [t_n, t_{n+1}] \tag{19}$$

$$\tau_n(t_n) = \tau_{n+1}(t_n). \tag{20}$$

To do so, observe from the same argument as in (14) that

$$\tau'_{n+1}(t) = \left(1 - \frac{\phi\left[\int_{\tau_{n-1}^{-1}[\tau_n^{-1}(t)]}^{\tau_n^{-1}(t)} h(z)dz\right] h[\tau_n^{-1}(t)]}{\tau'_n[\tau_n^{-1}(t)]}\right) + \phi\left[\int_{\tau_n^{-1}(t)}^{t} h(z)dz\right] h(t). \qquad (21)$$

But, by setting $k = n$ and evaluating hypothesis (17) at the point $\tau_n^{-1}(t) \in [t_{n-1}, t_n]$, it follows that the first term is again positive, so that τ_{n+1} satisfies (19). Moreover, by (17) and the hypothesized invertibility of the functions $\tau_k, k = 2, \ldots, n$, it follows that $\tau_n(t_{n-1}) = t_n \Rightarrow t_{n-1} = \tau_n^{-1}(t_n)$, and $t_n = \tau_{n-1}(t_{n-1}) \Rightarrow \tau_{n-1}^{-1}(t_n) = t_{n-1}$ which together with (18) yields

$$\begin{aligned}
\tau_{n+1}(t_n) &= t_n + \int_{\tau_n^{-1}(t_n)}^{t_n} \phi\left[\int_{\tau_{n-1}^{-1}(s)}^{s} h(z)dz\right] h(s)ds + (0) + \beta && (22)\\
&= t_n + \int_{t_{n-1}}^{t_n} \phi\left[\int_{\tau_{n-1}^{-1}(s)}^{s} h(z)dz\right] h(s)ds\beta && (23)\\
&= t_n + (0) + \int_{t_{n-1}}^{t_n} \phi\left[\int_{\tau_{n-1}^{-1}(s)}^{s} h(z)dz\right] h(s)ds\beta \\
&= \tau_n(t_n)
\end{aligned}$$

so that (20) also holds.

Finally, if we let $t_0 = 0$, then it follows by induction that the exit function, $\tau : R_+ \to R_+$, must be continuous and increasing on each interval $[t_n, t_{n+1}], n \geq 0$ if it is regular. But, since (13) and (23) imply that the combined interval, $\cup_{n\geq 0}[t_n, t_{n+1}]$, is connected, and since (8), (12), and (18) also imply that $t_{n+1} - t_n = \tau_{n+1}(t_n) - t_n \geq \beta > 0$, for all $n \geq 0$, we may conclude that $R_+ = \cup_{n\geq 0}[t_n, t_{n+1}]$, and thus that τ is everywhere continuous and increasing on R_+. Q.E.D.

Of course, with this result, we also know that if, by convention, we let $\tau^{-1}(t) = 0$ for all $t \in [0, \beta]$ then it follows that $T(t) = [\tau^{-1}(t), t]$ and hence that (2) can be written as

$$D(t) = \int_{\tau^{-1}(t)}^{t} \phi[x(s)]h(s)ds + \beta. \qquad (24)$$

3 Examples

It is, unfortunately, somewhat difficult to have much intuition about these link delay functions. Hence, in this section we explore two numerical examples.

3.1 Linear Delay Operators

One of the easiest cases to consider is the one in which $Phi[x(t)]$ is constant [this was used in Friesz et al. (1993) to develop a simultaneous route and departure-time choice equilibrium model]. This leads to a function of the following form:

$$D[x(t)] = \gamma x(t) + \beta. \tag{25}$$

where $\gamma, \beta > 0$.

Loosely speaking, γ can be viewed as the inverse of the service rate of a deterministic queue, and hence the link can be thought of as having a segment with a constant travel time of β and a segment (at the beginning or end of the link) with a queueing time of $\gamma x(t)$. However, this is not a strictly accurate interpretation of this delay function since the delay is a function of everyone on the link at time t, not everyone that is in the queue at time t (if the queue is at the upstream end of the link) or everyone that will be in the queue at time $t + \beta$ (if the queue is at the downstream end of the link).

Hence, in order to gain a better understanding of this type of link delay function we now consider an example in which we assume that the vehicles enter at a constant rate of α over some interval $[0, t_l]$. With this assumption, it is easy to determine x, D, and τ over the interval $[0, \beta]$ since no vehicles exit during that interval. In particular, it follows that:

$$\begin{aligned} x(t) &= \alpha t \quad t \in [0, \beta] & (26) \\ D(t) &= \int_0^t \gamma \alpha ds + \beta = t\gamma\alpha + \beta \quad t \in [0, \beta] & (27) \\ \tau(t) &= t + D(t) = t(1 + \gamma\alpha) + \beta \quad t \in [0, \beta] & (28) \end{aligned}$$

Further, since $\tau(\beta) = 2\beta + \beta\gamma\alpha$, it also follows that:

$$\tau^{-1}(t) = \frac{t-\beta}{(1+\gamma\alpha)} \quad t \in [\beta, 2\beta + \beta\gamma\alpha]. \tag{29}$$

Now, using (29), we can determine x, D, and τ over the interval $[\beta, 2\beta + \beta\gamma\alpha]$. In particular:

$$\begin{aligned}
x(t) &= \int_{\tau^{-1}(t)}^{t} \alpha ds = \alpha t - \frac{\alpha(t-\beta)}{(1+\gamma\alpha)} \quad t \in [\beta, 2\beta + \beta\gamma\alpha] && (30)\\
D(t) &= \int_{\tau^{-1}(t)}^{t} \gamma\alpha ds + \beta = t\gamma\alpha - \frac{\gamma\alpha(t-\beta)}{(1+\gamma\alpha)} + \beta \quad t \in [\beta, 2\beta + \beta\gamma\alpha] && (31)\\
\tau(t) &= t + D(t) = t(1+\gamma\alpha - \frac{\gamma\alpha}{1+\gamma\alpha}) + \frac{\gamma\alpha\beta}{1+\gamma\alpha} + \beta \quad t \in [\beta, 2\beta + \beta\gamma\alpha] && (32)
\end{aligned}$$

Thus, letting $t_3 = \tau(\beta, 2\beta + \beta\gamma\alpha) = 3\beta + 3\beta\gamma\alpha + \beta\gamma^2\alpha^2 + \frac{\beta\gamma^2\alpha^2}{1+\gamma\alpha}$, it follows that:

$$\tau^{-1}(t) = \frac{t - \beta - \frac{\gamma\alpha\beta}{1+\gamma\alpha}}{1+\gamma\alpha - \frac{\gamma\alpha}{1+\gamma\alpha}} \quad t \in [2\beta + \beta\gamma\alpha, t_3]. \tag{33}$$

Finally, proceeding for one more interval, it follows from (33) that:

$$\begin{aligned}
x(t) &= \int_{\tau^{-1}(t)}^{t} \alpha ds = \alpha t - \frac{\alpha(t-\beta) - \frac{\gamma\alpha^2\beta}{1+\gamma\alpha}}{1+\gamma\alpha - \frac{\gamma\alpha}{1+\gamma\alpha}} \quad t \in [2\beta + \beta\gamma\alpha, t_3] && (34)\\
D(t) &= \int_{\tau^{-1}(t)}^{t} \gamma\alpha ds + \beta = t\gamma\alpha - \gamma\frac{\alpha(t-\beta) - \frac{\gamma\alpha^2\beta}{1+\gamma\alpha}}{1+\gamma\alpha - \frac{\gamma\alpha}{1+\gamma\alpha}} + \beta \quad t \in [2\beta + \beta\gamma\alpha, t_3] && (35)\\
\tau(t) &= t + D(t) \quad t \in [2\beta + \beta\gamma\alpha, t_3]. && (36)
\end{aligned}$$

Note that these results imply that the exit rate, $g(t)$, is given by:

$$\begin{aligned}
g(t) &= 0 \quad t \in [0, \beta] && (37)\\
g(t) &= \frac{\alpha}{1+\gamma\alpha} \quad t \in [\beta, 2\beta + \beta\gamma\alpha] && (38)\\
g(t) &= \frac{\alpha}{1+\gamma\alpha - \frac{\gamma\alpha}{1+\gamma\alpha}} \quad t \in [2\beta + \beta\gamma\alpha, t_3]. && (39)
\end{aligned}$$

Thus, the exit rate is increasing over time even though vehicles are entering at a steady rate. This is because, while they are travelling more slowly, they are also getting close together.

We now consider specific values for these parameters. In particular, we consider a small one-lane road (0.5 miles long) with vehicles entering at a rate of 2000 per hour for 1.5 hours (from time 0 to time 1.5). For this example, we assume that $\gamma = 0.0003$, $\beta = 0.5$ hours, and that the free-flow speed is 60 miles per hour. The resulting plots of x and D are shown in Figure 1.

This figures display all of the expected properties. x and D both build over time and then decline. Conversely, the velocity, u, starts at 60mph and then falls to about 30mph before it begins to climb.

In order to better understand how this relates to traditional models of traffic flow, we also wanted to calculated the space mean, which is given by:

$$\overline{u}(t) = \int_{\tau^{-1}(t)}^{t} u(s)ds/x(t). \tag{40}$$

Given the complexity of this expression, these values were calculated numerically and are presented in Figure 2. In this figure, we plot the space mean speed against the "flow" using the fundamental relationship of traffic flow. In particular, letting the density be given by $k(t) = x(t)/(\beta 60)$, it follows that the flow is given by $q(t) = \overline{u}(t)k(t)$.

This speed-flow curve has the characteristic backward-bending shape that has been frequently observed empirically.

3.2 A Nonlinear Delay Operators

In order to better understand this class of delay functions we also consider an example in which $\Phi[x(t)] = \gamma x(t)$. Proceeding as before, it is relatively easy to derive x, D, and τ for $t \in [0, \beta]$:

$$x(t) = \alpha t \quad t \in [0, \beta] \tag{41}$$

$$D(t) = \int_0^t \gamma x(s) \alpha ds + \beta = 0.5\gamma\alpha^2 t^2 + \beta \quad t \in [0, \beta] \tag{42}$$

$$\tau(t) = t + D(t) = t + 0.5\gamma\alpha^2 t^2 + \beta \quad t \in [0, \beta] \tag{43}$$

Hence, it follows that:

$$\tau^{-1}(t) = \frac{[4 + 8\alpha\gamma^2(t-\beta)]^{1/2}}{2\gamma\alpha^2} - \frac{1}{\gamma\alpha^2} \quad t \in [\beta, \tau(\beta)] \tag{44}$$

Unfortunately, it is difficult to get much past this first interval analytically, since the expressions for D become unwieldly. hence, to explore this class of functions we proceeded numerically. In particular, we considered the same example as above, but now with $\gamma = 0.0000005$. The results are plotted in Figures 3 and 4.

It is interesting to note that these curves have the same basic shape as in the linear case.

References

[1] Ben-Akiva, M., A. de Palma, and I. Kaysi (1991) "Dynamic Network Models and Driver Information Systems", *Transportation Research*, 25A, 251-266.

[2] Bernstein, D., T.L. Friesz, R.L. Tobin, and B-W. Wie (1993) "Variational Control Models of Dynamic Network Traffic Equilibrium", *Proceedings of the 12th International Symposium on Transportation and Traffic Theory*, Elsevier, Amsterdam.

[3] Boardman, A.E. and L.B. Lave (1977) "Highway Congestion and Congestion Tolls", *Journal of Urban Economics*, Vol. 4, pp. 340-359.

[4] Cascetta, E. (1991) "A Day-to-Day and Within-Day Dynamic Stochastic Assignment Model, *Transportation Research*, Vol. 25A, pp. 277-291.

[5] Chang, G-L. and H.S. Mahmassani (1988) "Travel Time Prediction and Departure Time Adjustment Behavior Dynamics in a Congested Traffic System", *Transportation Research*, Vol. 22B, pp. 217-232.

[6] Drissi-Kaitouni, O. (1990) "A Model for the Dynamic Traffic Assignment Problem", Centre de Recherche sur les Transports, Publication No. 702, Universite de Montreal.

[7] Friesz, T.L., D. Bernstein, T.E. Smith, R.L. Tobin, and B.-W. Wie (1993) "A Variational Inequality Formulation of the Dynamic Network User Equilibrium Problem", *Operations Research*, Vol. 41, 1993, pp. 179-191.

[8] Janson, B.N. (1991) "Dynamic Traffic Assignment for Urban Road Networks", *Transportation Research*, Vol. 25B, pp. 143-161.

[9] Kraft, G. and M. Wohl (1967) "New Directions for Passenger Demand Analysis and Forecasting", *Transportation Research*, Vol. 1, pp. 205-230.

[10] Mosher, W.W. (1963) "A Capacity Restraint Algorithm for Assigning Flow to a Transport Network", *Highway Research Record*, Vol. 6, pp. 41-70.

[11] Ran, B., D. Boyce. and L.J. LeBlanc (1993) *Operations Research*, Vol. 41.

[12] Smith, M.J. and M.O. Ghali (1990) "The Dynamics of Traffic Assignment and Traffic Control: A Theoretical Study", *Transportation Research*, Vol. 24B, pp. 409-422.