Operations on Relations

(CS 227 class notes prepared by Ramon A. Mata-Toledo)

Operations on relations

As we previously defined, a binary relation between two nonempty sets A and B (in that order) is nothing more than a subset of the Cartesian product of A and B (n that order). Since relations are sets they are susceptible to the ordinary set operations. For example, given two relations R1 and R2 we can form the union, intersection, and difference of these two relations. The previously mentioned operations are defined as indicated below.

Definition No. 1 (Union of two relations)

Given two relations R_1 and R_2, we will define the Union of these relations, denoted by $R_1 \cup R_2$, as

$$R_1 \cup R_2 = \{(a,b) / (a,b) \in R_1 \vee (a,b) \in R_2\}$$ ■

Definition No. 2 (Intersection of two relations)

Given two relations R_1 and R_2, we will define the Intersection of these relations, denoted by $R_1 \cap R_2$, as

$$R_1 \cap R_2 = \{(a,b) / (a,b) \in R_1 \wedge (a,b) \in R_2\}$$ ■

Note No. 1

Notice that when we talk about the operations of Union or Intersection the order of the relations is immaterial.

Definition No. 3 (Difference of two relations)

Given two relations R_1 and R_2, we will define the difference of relations R_1 and R_2 (in that order), denoted by $R_1 - R_2$, as follows:

$$R_1 - R_2 = \{(a,b) / (a,b) \in R_1 \wedge (a,b) \notin R_2\}$$ ■

Note No. 2

Notice that when we define the difference of two relations the order in which they are mentioned matter. In general, $R_1 - R_2$ will be different than $R_2 - R_1$. ■

Definition No. 4 (The inverse relation of given relation)

Given a relation R, we will define the inverse of this relation, denoted by R^{-1}, as follows:

$$R^{-1} = \{(a,b) / (a,b) \in R\}$$ ■

Example No. 1 (from Rosen)

Given R_1 = {(1,2), (2,3), (3,4)} and R_2 = {(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) }

Find

a) $R_1 \cup R_2 =$

b) $R_1 \cap R_2 =$

c) $R_1 - R_2 =$

d) $R^{-1} =$

e) $R^{-2} =$

f) $R^{-1} \cap R^{-2} =$

g) $R^{-1} \cup R^{-2} =$

Definition No. 5 (composition of two relations)

Given two relations R_1 and R_2 we will define the composite of R_1 and R_2 (in that order) as follows:

$$R_1 \circ R_2 = \{(a,c) / (a,b) \in R_1 \wedge (b,c) \in R_2\} \qquad \blacksquare$$

Example No. 2

Given $R_1 = \{(1,2), (2,3), (3,4)\}$ and $R_2 = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ find the composite relation $R_1 \circ R_2$ and $R_2 \circ R_1$

Definition No. 6 (first and second projection of a graph)

Given a relation R, we will call the first projection of R, denoted by, $\Pr_1 R$, to the set

$$\Pr_1 R = \{a / (a,b) \in R\} \qquad \blacksquare$$

Likewise, we will call the second projection of R, denoted by, $\Pr_2 R$, to the set

$$\Pr_2 R = \{b / (a,b) \in R\} \qquad \blacksquare$$

Example No. 3

Given $R_1 = \{(1,2), (2,3), (3,4)\}$ and $R_2 = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$ find the following sets:

a) $\Pr_1 R_1$
b) $\Pr_2 R_1$
c) $\Pr_1 R_2$
d) $\Pr_2 R_2$
e) $\Pr 1R1 \bigcap \Pr_2 R_2$
f) $\Pr 1R1 \bigcup \Pr_2 R_2$

References

The order of presentation follows Discrete Mathematics and Its Applications by Rosen. Exercises or examples have been adopted from the remaining books listed in this section.

1) Discrete Mathematics and Its Applications by K. H. Rosen. 6^{th} Ed. McGraw-Hill, 2007
2) Introduccion a la Teoria de Conjuntos by L. Oubiña. Editorial Universitaria de Buenos Aires,1965. (Spanish version only)
3) Sets by W.W. Fairchild and C. Ionescu Tulcea. W.B Saunders Company, 1970.