Relations
(CS 228 class notes prepared by Ramon A. Mata-Toledo)

Reading Assignments (Rosen)

- Read Section 2.1 pages 117-118 on your textbook.
- Read Section 8.1 pages 519-527 on your textbook

Definition No. 1 (**Cartesian product of two sets**)

Given two nonempty sets *A* and *B* we will call the Cartesian product of *A* and *B* (in this order) to the following set:

$$A \times B = \{(a,b) \,/\, a \in A \wedge b \in B\}$$

The elements (*a*,*b*) of the Cartesian product are called ordered pairs. The elements *a* and *b* are generally called the first coordinate (or component) and the second coordinate (or component) respectively.

Example No. 1

A = {1, 2} and B = {a, b} then

A x B = { (1,a), (1,b), (2, a), (2,b)} and B x A = {(a,1), (1,b), (2,a), (2,b)}.

The ordered pairs of the set A x B are : (1,a), (1,b), (2, a), and (2,b)

The ordered pairs of the set B x A are: (a,1), (1,b), (2,a), and (2,b)

Notice that (1,a) $\in$ AxB and (1,a) $\notin$ BxA. Therefore, AxB $\neq$ BxA (why?)

In general, for any two sets A and B we have that A x B $\neq$ B x A

Note No. 1

Ordered pairs can be formally defined as follows:

$$(a,b) = \{\{a\}, \{a,b\}\}$$

From this definition we can observe that if $a \neq b$ then $(a,b) \neq (b,a)$. This justifies the previous assertion that, in general, A x B $\neq$ B x A. ■

The next theorem allows us to define the equality of ordered pairs.

Theorem No. 1 (equality of ordered pairs)

Two ordered pairs (a,b) and (c,d) are equal their respective coordinates are equal. That is,

$$(a,b) = (c,d) \leftrightarrow a = b \wedge c = d$$ ■

Note No. 2

If A = B then the Cartesian product A x B is generally denoted by A^2. ■

Exercise No. 1

Given A = {0, 1, 2} and B = {x, y, z} what are the elements of A x B and B x A. How many elements are in the Cartesian product of each set? ■

Definition No. 2 (Binary relations between two given sets)

Let *A* and *B* be two nonempty sets. A binary relation *R* from *A* to *B* (in that order) is a nonempty subset of the Cartesian product of *A* and *B*.

If $R \subset AxB$ and $(x, y) \in R$ then x is said to be R-related to y. We can denote this by writing xRy. ■

Example No. 2

Given the Cartesian product A x B = { (1,a), (1,b), (2, a), (2,b)}. Let's consider R = {(1,a), (1,b), (2, a)}. Then we can say that

1Ra since (1,a) $\in$ R

1Rb since (1,b) $\in$ R

2Ra since (2,a) $\in$ R

Notice that (2,b) $\notin$ R. In this case we can write $2 \not R b$. We read $2 \not R b$ as "2 is not R-related to b."

From Example No. 2 we can generalize and say that $x \not R y \leftrightarrow$ (x,y) $\notin$ R. ■

Note No. 3

Since any set is a subset of itself, in particular, we have that AxB $\subset$ AxB. Therefore, the Cartesian product of two given set also defines a binary relationship between the sets. ■

Definition No. 3 (Relation on a set A)

Any subset of the Cartesian product of AxA is called a relation on the set A or just a relation on A. ■

Exercise No. 2

Given the set A = {1,2,3}. State if the following sets are relations on A and explain why.

{(1,1), (2,2),(3,3)}

{(1,2), (2,3), (3,1),(3,2)

{(1,1)}

{1, 2}

{1, 2, 3}

■

A relation on a set A can be defined by some conditions that the elements of the ordered pairs need to satisfy. This is illustrated in the following example.

Example No. 3

Let A = {1, 2, 3, 4} Find the elements of the relation on A defined as follows: $R = \{(a,b) / a < b\}$.

AxA = {(1,1), (1,2), (1,3), (2,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)}

Then R = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)} ■

Properties of Relations on a set A.

Let A be a set (A≠∅) and R a relation on A.

Definition No. 4 (reflexive relation)

We will say that a relation R is reflexive if and only if for every element a∈A we have that aRa. That is, (a,a) ∈ R for any element a∈A. ■

Definition No. 5 (symmetric relation)

We will say that a relation R is symmetric if and only if aRb implies bRa for any element a,b ∈ A. In other words, a relation is symmetric whenever (a,b)∈R we also have (b,a) ∈ R. ■

Definition No. 6 (transitive relation)

We will say that a relation R is transitive if and only if aRb and bRc implies aRc for any element a,b,c∈A. In other words, if (a,b)∈R and (b,c)∈R then (a,c)∈R for all elements a,b,c∈ A. ■

Example No. 4 (from Rosen)

Given A = {1, 2, 3, 4} and the relations shown below find which relations are a) reflexive b) symmetric and c) transitive.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}

R_2 = {(1,1), (1,2), (1,1)}

R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}

R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}

R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}

R_6 = {(3,4)}

References

The main reference is Discrete Mathematics and Its Applications by Rosen. However, exercises or examples have been adopted from the remaining books listed in this section.

1) Discrete Mathematics and Its Applications by K. H. Rosen. 6th Ed. McGraw-Hill, 2007
2) Introduccion a la Teoria de Conjuntos by L. Oubiña. Editorial Universitaria de Buenos Aires,1965. (Spanish version only)
3) Discrete Mathematics with Applications by S. Epp. 3rd Ed. Brook/Cole, 2004.

4) 2000 Solved problems in Discrete Mathematics by S. Lipschutz and M. C. Lipson. Schaum's Solved Problem Series. 1992.