1. How many bit strings of length seven either begin with two 0s or end with three 1s?

\[ A = \{00\ldots\} \quad B = \{\ldots111\} \]

\[ |A| = 2^5 = 32 \]

\[ |B| = 2^4 = 16 \]

\[ |A \cap B| = 2^2 = 4 \]

\[ |A \cup B| = |A| + |B| - |A \cap B| = 32 + 16 - 4 = 44 \]

2. Thirteen people on a softball team show up for a game
   a) How many ways are there to choose ten players to take the field?

\[ \binom{13}{10} = 286 = \frac{13!}{10! \cdot 3!} \]

b) How many ways are there to assign the ten positions by selecting players from the 13 people who show up?

\[ P_{10} = 1,037,836,800 = \frac{13!}{(13 - 10)!} \]

c) Of the 13 people who show up, three are women. How many ways are there to choose ten players to take the field if at least one of these players must be a woman?

\[ 1 = \binom{13}{10} - \# \text{ of ways to select a team with no women} \]

\[ = \binom{13}{10} - \binom{10}{10} = 286 - 1 = 285 \]

\[ \text{OR} \]

\[ \binom{3}{1} \times \binom{10}{9} + \binom{3}{2} \times \binom{10}{8} + \binom{3}{3} \times \binom{10}{7} \]

Notice we need at least (not exactly) one woman.

i.e. one W or two Ws or three Ws.
3. Prove that if $n$ and $k$ are positive integers, then \[
\binom{n+1}{k} = \frac{n+1}{k} \binom{n}{k-1}
\]

\[
\begin{align*}
R.H.S &= \frac{n+1}{k} \binom{n}{k-1} = \frac{n+1}{k} \cdot \frac{n!}{(k-1)! (n-k+1)!} \\
&= \frac{(n+1)(n)!}{k \cdot (k-1)! (n+1-k)!} = \frac{(n+1)!}{k! (n+1-k)!} = \binom{n+1}{k} \\
&= L.H.S.
\end{align*}
\]

4. Use the Principle of Mathematical Induction to prove that $2n + 3 \leq 2^n$ for all $n \geq 4$.

**Base Step** @ $n=4$

\[2 \cdot 4 + 3 \ ? \ 2^4 \]

\[11 \ \leq \ 16 \quad \therefore p(4) \text{ is true}
\]

**Inductive Step** To Show $p(k) \Rightarrow p(k+1)$

where $p(k): 2k + 3 \leq 2^k$, $k \geq 4$

$p(k+1): 2(k+1) + 3 \leq 2^{k+1}$

Consider

\[
2(k+1) + 3 = (2k+3) + 2 \\
\leq 2^k + 2 \\
\leq 2^k + 2^k \\
\leq 2^k + 2^k, \text{ by } p(k)
\]

\[
\therefore 2(k+1) + 3 < 2^{k+1}
\]

\[
\therefore p(k+1) \text{ is true.}
\]
5. What is the coefficient of $x^8y^9$ in the expansion of $(3x + 2y)^{17}$?

\[
\binom{17}{8}(3x)^8(2y)^9
\]

\[
\binom{17}{8} \times 3 \times 2
\]

\[
= 24310 \times 6561 \times 512 = 81662,924,920
\]

which is the same as

\[
\binom{17}{9} \times 3 \times 2
\]

6. The following is a relation on the set \{ a, b, c, d \}:

\[ R = \{ (a, b), (b, c), (c, b), (d, c), (d, a) \} \]

i) Draw the directed graph of this relation

\[
\begin{align*}
\text{a} & \rightarrow \text{b} \\
\text{b} & \rightarrow \text{c} \\
\text{c} & \rightarrow \text{b} \\
\text{d} & \rightarrow \text{c} \\
\end{align*}
\]

ii) Now, draw the transitive closure of $R$ by completing the figure below

\[
\begin{align*}
\text{a} & \rightarrow \text{b} \\
\text{b} & \rightarrow \text{c} \\
\text{c} & \rightarrow \text{b} \\
\text{d} & \rightarrow \text{c} \\
\end{align*}
\]

iii) and the reflexive closure of $R$

\[
\begin{align*}
\text{a} & \rightarrow \text{b} \\
\text{b} & \rightarrow \text{c} \\
\text{c} & \rightarrow \text{b} \\
\text{d} & \rightarrow \text{c} \\
\text{a} & \rightarrow \text{a} \\
\text{b} & \rightarrow \text{b} \\
\text{c} & \rightarrow \text{c} \\
\text{d} & \rightarrow \text{d} \\
\end{align*}
\]

iv) and the symmetric closure of $R$

\[
\begin{align*}
\text{a} & \rightarrow \text{b} \\
\text{b} & \rightarrow \text{c} \\
\text{c} & \rightarrow \text{b} \\
\text{d} & \rightarrow \text{c} \\
\text{a} & \rightarrow \text{b} \\
\text{b} & \rightarrow \text{c} \\
\text{c} & \rightarrow \text{b} \\
\text{d} & \rightarrow \text{c} \\
\end{align*}
\]

Notice that any closure contains the original relation in addition to possibly more ordered pairs (i.e. arcs)

My Best Wishes
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