1.8 FUNCTIONS

DEF: Let A and B be sets. A ***function*** f (or more completely, $f : A \to B$) is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$, called the ***value*** of f at a.

We also say that $f : A \to B$ is a ***mapping*** from ***domain*** A to ***codomain*** B.

$f(a)$ is called the ***image of the element*** a, and the element a is called a ***preimage*** of $f(a)$.

The set $\{a \mid f(a) = b\}$ is called the ***preimage set*** of b. NOTATION: $f^{-1}(b)$.

DEF: The set $\{b \in B \mid (\exists a \in A)[f(a) = b]\}$ is called the ***image of the function*** $f : A \to B$.

DEF: The word ***range*** is commonly used to mean the image.

DEF: A function is ***discrete*** if its domain and codomain are finite or countable (indexed by $\mathcal{Z}$).

Example 1.8.1: Some functions from $\mathcal{R}$ to $\mathcal{Z}$.

(1) ***floor*** $\lfloor x \rfloor = \max\{k \in \mathcal{Z} \mid k \leq x\}$ image $= \mathcal{Z}$

(2) ***ceiling*** $\lceil x \rceil = \min\{k \in \mathcal{Z} \mid k \geq x\}$ im $= \mathcal{Z}$

(3) ***sign*** $\sigma(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & x > 0 \end{cases}$ im $= \{-1, 0, +1\}$

Example 1.8.2: Seq of functions from $\mathcal{R}$ to $\mathcal{R}$.

falling powers $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$.

$$7^{\underline{3}} = 7 \cdot 6 \cdot 5 = 210 \qquad \left(\frac{3}{2}\right)^{\underline{3}} = \frac{3}{2} \cdot \frac{1}{2} \cdot \left(\frac{-1}{2}\right) = \frac{-3}{8}$$

Example 1.8.3: Functions in computation.

(1) ***C compiler*** maps the set of ASCII strings to the boolean set.

(2) The ***halting function*** maps the set of C programs to the boolean set, assigns TRUE iff this program will always halt eventually, no matter what input is supplied at run time.

Theorem 1.8.1. *The halting function cannot be represented by a C program.* $\diamondsuit$ *(CS W3261)*

REPRESENTATION of DISCRETE FUNCTIONS

DEF: The $n \times 2$ ***array representation*** of a discrete function is a table with two columns. The left column contains every element of the domain. The second entry in each row is the image of the first entry.

DEF: The ***(full) digraphic representation*** of a discrete function is a diagram with two columns of dots. The left column contains a dot for every element of the domain, and the right entry contains a dot for every element of the codomain. From each domain dot an arrow is drawn to the codomain dot representing its image.

Example 1.8.4: Representing a function.

ONE-TO-ONE and ONTO FUNCTIONS

DEF: A function $f : A \to B$ is ***one-to-one*** if for every $b \in B$, there is at most one $a \in A$ such that $f(a) = b$.

Proposition 1.8.2. *A discrete function is one-to-one if and only if in its digraphic representation, no codomain dot is at the head of more than one arrow.* ♢

DEF: A function $f : A \to B$ is ***onto*** if for every $b \in B$, there is at least one $a \in A$ such that $f(a) = b$.

Proposition 1.8.3. *A discrete function is onto if and only if in its digraphic representation, every codomain dot is at the head of at least one arrow.* ♢

Example 1.8.5: The grading function of Example 1.8.4 is neither one-to-one or onto.

BIJECTIONS

DEF: A ***bijection*** is a function that is one-to-one and onto.

DEF: Let $f : A \to B$ be a bijection. The ***inverse function*** $f^{-1} : B \to A$ is the rule that assigns to each $b \in B$ the unique element $a \in A$ such that $f(a) = b$.

Example 1.8.6: The function

$$\{1 \mapsto b, 2 \mapsto c, 3 \mapsto a\}$$

is a bijection. Its inverse is the function

$$\{a \mapsto 3, b \mapsto 1, c \mapsto 2\}$$

DEF: A ***permutation*** is a bijection whose domain and codomain are the same set.

Example 1.8.7: The function

$$\{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$$

is a permutation. Its inverse is the permutation

$$\{1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2\}$$