1.8 FUNCTIONS

DEF: Let $A$ and $B$ be sets. A function $f$ (or more completely, $f : A \to B$) is a rule that assigns to each element $a \in A$ exactly one element $f(a) \in B$, called the value of $f$ at $a$.

We also say that $f : A \to B$ is a mapping from domain $A$ to codomain $B$.

$f(a)$ is called the image of the element $a$, and the element $a$ is called a preimage of $f(a)$.

The set $\{a \mid f(a) = b\}$ is called the preimage set of $b$. NOTATION: $f^{-1}(b)$.

DEF: The set $\{b \in B \mid (\exists a \in A)[f(a) = b]\}$ is called the image of the function $f : A \to B$.

DEF: The word range is commonly used to mean the image.

DEF: A function is discrete if its domain and codomain are finite or countable (indexed by $\mathbb{Z}$).
Example 1.8.1: Some functions from $\mathbb{R}$ to $\mathbb{Z}$.

1. **floor** $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$  
   $\text{image} = \mathbb{Z}$

2. **ceiling** $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$  
   $\text{im} = \mathbb{Z}$

3. **sign** $\sigma(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
+1 & \text{if } x > 0 
\end{cases}$  
   $\text{im} = \{-1, 0, +1\}$

Example 1.8.2: Seq of functions from $\mathbb{R}$ to $\mathbb{R}$.

**falling powers**  
$x^n = x(x-1)\cdots(x-n+1)$.

$7^3 = 7 \cdot 6 \cdot 5 = 210$  
$\left(\frac{3}{2}\right)^3 = \frac{3}{2} \cdot \frac{1}{2} \cdot \left(\frac{-1}{2}\right) = \frac{-3}{8}$

Example 1.8.3: Functions in computation.

1. **C compiler** maps the set of ASCII strings to the boolean set.

2. The **halting function** maps the set of C programs to the boolean set, assigns TRUE iff this program will always halt eventually, no matter what input is supplied at run time.

**Theorem 1.8.1.** The halting function cannot be represented by a C program. \hspace{1cm} $\blacklozenge$ (CS W3261)
REPRESENTATION of DISCRETE FUNCTIONS

DEF: The $n \times 2$ array representation of a discrete function is a table with two columns. The left column contains every element of the domain. The second entry in each row is the image of the first entry.

DEF: The (full) digraphic representation of a discrete function is a diagram with two columns of dots. The left column contains a dot for every element of the domain, and the right entry contains a dot for every element of the codomain. From each domain dot an arrow is drawn to the codomain dot representing its image.

Example 1.8.4: Representing a function.
**ONE-TO-ONE and ONTO FUNCTIONS**

**DEF:** A function $f : A \to B$ is **one-to-one** if for every $b \in B$, there is at most one $a \in A$ such that $f(a) = b$.

**Proposition 1.8.2.** A discrete function is one-to-one if and only if in its digraphic representation, no codomain dot is at the head of more than one arrow.  

**DEF:** A function $f : A \to B$ is **onto** if for every $b \in B$, there is at least one $a \in A$ such that $f(a) = b$.

**Proposition 1.8.3.** A discrete function is onto if and only if in its digraphic representation, every codomain dot is at the head of at least one arrow.

**Example 1.8.5:** The grading function of Example 1.8.4 is neither one-to-one or onto.
BIJECTIONS

DEF: A **bijection** is a function that is one-to-one and onto.

DEF: Let \( f : A \rightarrow B \) be a bijection. The **inverse function** \( f^{-1} : B \rightarrow A \) is the rule that assigns to each \( b \in B \) the unique element \( a \in A \) such that \( f(a) = b \).

**Example 1.8.6:** The function
\[
\{1 \mapsto b, 2 \mapsto c, 3 \mapsto a\}
\]
is a bijection. Its inverse is the function
\[
\{a \mapsto 3, b \mapsto 1, c \mapsto 2\}
\]

DEF: A **permutation** is a bijection whose domain and codomain are the same set.

**Example 1.8.7:** The function
\[
\{1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}
\]
is a permutation. Its inverse is the permutation
\[
\{1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2\}\]