1.5 METHODS OF PROOF

Some forms of argument ("valid") never lead from correct statements to an incorrect conclusion. Some other forms of argument ("fallacies") can lead from true statements to an incorrect conclusion.

DEF: An ***axiom*** is a statement that is assumed to be true, or in the case of a mathematical system, is used to specify the system.

DEF: A ***mathematical argument*** is a list of statements. Its last statement is called the ***conclusion***.

DEF: A ***logical rule of inference*** is a method that depends on logic alone for deriving a new statement from a set of other statements.

DEF: A ***mathematical rule of inference*** is a method for deriving a new statement that may depend on inferential rules of a mathematical system as well as on logic.

VALID ARGUMENTS

DEF: A ***logical argument*** consists of a list of (possibly compound) propositions called premises and a single proposition called the conclusion.

Example 1.5.1: A Logical Argument

If I dance all night, then I get tired.

I danced all night.

Therefore I got tired.

Logical representation of underlying variables:

p: I dance all night. q: I get tired.

Logical analysis of argument:

$p \rightarrow q$	premise 1
p	premise 2
q	conclusion

DEF: A form of logical argument is ***valid*** if whenever every premise is true, the conclusion is also true. A form of argument that is not valid is called a ***fallacy***.

We shall see why the argument above is valid.

This form of argument is called ***modus ponens***.

$p \to q$	premise 1
p	premise 2
q	conclusion

Operational Method of Validation

Step 1. Form a truth table in which the premises are columns, and the conclusion is the last column.

Step 2. Star every row in which all the premises are true.

Step 3. Declare the argument to be valid if every starred row has a T in its last column (the conclusion column).

Having once verified modus ponens with truth tables, we need never question its validity again.

FALLACIES

Example 1.5.2: A Fallacy

If I dance all night, then I get tired.

I got tired.

Therefore I danced all night.

Logical form of argument:

$p \rightarrow q$	premise 1
q	premise 2
p	conclusion

Now for the validity check.

Row 3 indicates it is possible that even when all the premises are true, the conclusion can be false. Thus, this form of argument is a fallacy.

NOTORIOUS FALLACIES

In the ***fallacy of affirming the consequent***, one affirms the consequent of a conditional and concludes that the antecedent is true.

$$[(p \to q) \land q] \to p$$

Example 1.5.2 affirms the consequent.

In the ***fallacy of denying the antecedent***, one denies the antecedent of a conditional and concludes that the consequent is false.

$$[(p \to q) \land \neg p] \to \neg q$$

Example 1.5.3: Denying the Antecedent

She says: Even if you were the last man on earth, I would not marry you.

He thinks: But I am not the last man on earth, and that implies that she will marry me. I'll keep on trying.

Validity and Truth

(1) The conclusion of a valid argument might be false, if one or more of the premises is not true.

(2) The conclusion of a fallacy might be true.

(3) If the premises are correct, and if the argument is valid, then the conclusion is correct.

LOGICAL RULES of INFERENCE

TERMINOLOGY NOTE: A ***rule of inference*** is defined to be any valid argument. Typically, however, it is only called a ***valid argument*** unless it is frequently applied.

All of the following rules of inference can be confirmed with truth tables.

Modus Ponens.

$p \to q$	premise 1
p	premise 2
q	conclusion

Modus Tollens.

$p \to q$	premise 1
$\neg q$	premise 2
$\neg p$	conclusion

Addition.

$$\begin{array}{ll} \underline{\quad p \qquad\qquad} & \underline{\text{premise 1}} \\ p \vee q & \text{conclusion} \end{array}$$

Simplification.

$$\begin{array}{ll} \underline{p \wedge q \qquad} & \underline{\text{premise 1}} \\ \quad p & \text{conclusion} \end{array}$$

Disjunctive Syllogism.

$$\begin{array}{ll} p \vee q & \text{premise 1} \\ \underline{\quad \neg p \qquad} & \underline{\text{premise 2}} \\ \quad q & \text{conclusion} \end{array}$$

Hypothetical Syllogism.

$$\begin{array}{ll} p \rightarrow q & \text{premise 1} \\ \underline{q \rightarrow r \quad} & \underline{\text{premise 2}} \\ p \rightarrow r & \text{conclusion} \end{array}$$

MATHEMATICAL PROOFS (DIRECT)

DEF: A ***direct proof*** is a mathematical argument that uses rules of inference to derive the conclusion from the premises.

Example 1.5.4: Alt Proof of Disj Syllogism: by a chain of inferences.

$p \vee q$	**premise 1**
$q \vee p$	commutativity of $\vee$
$\neg\neg q \vee p$	double negation law
$\neg q \rightarrow p$	$A \rightarrow B \Leftrightarrow \neg A \vee B$
$\neg p$	**premise 2**
$\neg\neg q$	modus tollens
q	**conclusion** by dbl negation

Example 1.5.5: a theorem
The sum of two even numbers x and y is even.
Proof: (1) There exist numbers m and n such that $x = 2m$ and $y = 2n$ (by def of "even").

(2) Then $x + y = 2m + 2n$ (by substitution).

$= 2(m + n)$ (by left distrib)

which is even, by the defn of evenness. $\diamondsuit$

MATHEMATICAL PROOFS (INDIRECT)

DEF: An ***indirect proof*** uses rules of inference on the negation of the conclusion and on some of the premises to derive the negation of a premise. This result is called a ***contradiction***.

Example 1.5.6: a theorem
If x^2 is odd, then so is x.
Proof: Assume that x is even (neg of concl).

Say $x = 2n$ (defn of even).

Then $x^2 = (2n)^2$ (substitution)

$= 2n \cdot 2n$ (defn of exponentiation)

$= 2 \cdot 2n^2$ (commutativity of mult.)

which is an even number (defn of even)

which contradicts the premise that x^2 is odd. $\diamondsuit$

MATHEMATICAL PROOFS (by CASES)

DEF: A ***proof by cases*** uses the following rule of inference:

$p \to r$	premise 1
$q \to r$	premise 2
$p \vee q$	premise 3
r	conclusion

Example 1.5.7: a theorem
Let x be any integer. Then $x^2 + x$ is even.
Proof: setup for proof-by-cases inference
$p : x$ is even; $q : x$ is odd; $r : x^2 + x$ is even.

Verify premise 1. If x is even, then $x = 2n$, for some integer n. Hence, $x^2 + x = (2n)2 + 2n = 4n^2 + 2n$, which is even.

Verify premise 2. If x is odd, then $x = 2n+1$, for some n. Hence, $x^2 + x = (2n+1)2 + (2n+1) = (4n^2 + 4n + 1) + (2n + 1) = 4n^2 + 6n + 2$, which is even.

Verify premise 3: An arbitrary integer is either even or odd. $\diamondsuit$

PROVING QUANTIFIED ASSERTIONS

(1) To prove $(\forall x)[P(x)]$
Let x be an arbitrary (unrestricted) member of the universal set of context and prove that $P(x)$ is true.

Example: Show that $x^2 + x$ is even, for all x.

(2) To prove $(\exists x)[P(x)]$
Exhibit any member of the universe for which $P(x)$ is true. One example suffices.

Example: Show that 729 is a power of 3, that is, $(\exists n)[3^n = 729]$.

(3) To prove $\neg(\forall x)[P(x)]$
Exhibit any member of the universe for which $P(x)$ is false. One counterexample suffices.

Example: Show that 323 is not prime.

(4) To prove $\neg(\exists x)[P(x)]$
Let x be an arbitrary (unrestricted) member of the universal set of context and prove that $P(x)$ is false.

Example: Show that $\sqrt{2}$ is irrational, that is, $\neg(\exists p, q)[\sqrt{2} = \frac{p}{q}]$.

TERMINOLOGY

DEF: A ***mathematical proof*** is a list of statements in which every statement is one of the following:

(1) an axiom

(2) derived from previous statements by a rule of inference

(3) a previously derived *theorem*

Its last statement is called a ***theorem***.

TERMINOLOGY: There is a hierarchy of terminology that gives opinions about the importance of derived truths:

(1) A ***proposition*** is a theorem of lesser generality or of lesser importance.

(2) A ***lemma*** is a theorem whose importance is mainly as a key step in something deemed to be of greater significance.

(3) A ***corollary*** is a consequence of a theorem, usually one whose proof is much easier than that of the theorem itself.