1.3 PREDICATES AND QUANTIFIERS

DEF: Informally, a ***predicate*** is a statement about a (possibly empty) collection of variables over various domains. Its truth value depends on the values of the variables in their respective domains.

DEF: Formally, a ***predicate*** is a function from the cartesian product of the domains of the variables to the boolean set $\{T, F\}$.

Example 1.3.1: $x + 2 = 5$.

Example 1.3.2: $4x - 3y > 2x$.

DEF: The ***universal quantification (over x)*** of a predicate $P(x)$ is the predicate $(\forall x)[P(x)]$.

Example 1.3.3: $(\forall x)[x + 2 = 5]$.

Example 1.3.4: $(\forall x)[4x - 3y > 2x]$.

DEF: The ***existential quantification (over x)*** of a predicate $P(x)$ is the predicate $(\exists x)[P(x)]$.

Example 1.3.5: $(\exists x)[x + 2 = 5]$.

Example 1.3.6: $(\exists x)[4x - 3y > 2x]$.

Remark: Observe that the result of quantifying a predicate is still a predicate. Moreover, when propositional operators are applied to predicates, the results are predicates.

VARYING THE DOMAIN

Example 1.3.7: $(\forall x)[x^2 = 1]$ is FALSE over the domain of integers, but TRUE over the domain $\{-1, 1\}$.

Example 1.3.8: $(\exists x)[x^2 = -9]$ is FALSE over the integers, but TRUE over the domain of complex numbers.

CLASSROOM EXERCISE

Consider these two condition statements.

1. $(\forall x)[P(x)] \quad \rightarrow \quad (\exists x)[P(x)]$.

Over the domain of people, this would mean "If something is good for everybody, then it's good for somebody.".

2. $(\exists x)[P(x)] \quad \rightarrow \quad (\forall x)[P(x)]$.

Over the domain of people, this could mean "What's good for me is good for everybody.".

Try to think of a general property of a domain under which statement (1) is necessarily FALSE.

Try to think of a general property of a domain under which statement (2) is necessarily TRUE.

Hint: These general properties are based solely on the number of elements in the domain.

SCOPE of QUANTIFIERS

DEF: The ***scope*** of a quantifier is the clause to which it applies.

Example 1.3.9: Let x range over the integers.

$$P(x) : x > 2 \qquad Q(x) : x < 2$$

Compare these two non-equivalent propositions:

$A.(\exists x)[P(x) \leftrightarrow Q(x)] \quad B.(\exists x)[P(x)] \leftrightarrow (\exists x)[Q(x)]$

A is FALSE, but B is TRUE.

DEF: An ***unbound variable*** in a predicate is a variable not within the scope of any quantifier.

Example 1.3.10: x is an unbound variable.

$$x + 4 > 2$$

Example 1.3.11: x is an unbound variable.

$$(\forall y)[2x + 3y = 7]$$

Remark: A predicate with no unbound variables is a proposition.

NEGATION with QUANTIFIERS

p: There exists some input data for which this program will crash.

$\neg p$: No matter what input data you supply to this program, it will not crash.

Rule 1: $\neg(\exists x)[P(x)] \Leftrightarrow (\forall x)[\neg P(x)]$

Rule 2: $\neg(\forall x)[P(x)] \Leftrightarrow (\exists x)[\neg P(x)]$

CLASSROOM EXERCISE

On a New Jersey Transit commuter run, the conductor announces:

At the next stop, all doors will not be open.

Express this in symbolic logic.
Explain what his words mean.
What words accurately express what he probably intended?