1.1 LOGIC

Mathematics is used to predict empirical reality, and is therefore the foundation of engineering. Logic gives precise meaning to mathematical statements.

PROPOSITIONS

DEF: A ***proposition*** is a statement that is either true (T) or false (F), but not both.

Example 1.1.1:

$1 + 1 = 2$. (T)

$2 + 2 = 5$. (F)

Example 1.1.2: A fact-based declaration is a proposition, even if no one knows whether it is true.

11213 is prime.

1 is prime.

There exists an odd perfect number.

Example 1.1.3: Logical analysis of rhetoric begins with the modeling of fact-based natural language declarations by propositions.

Portland is the capital of Oregon.

Columbia University was founded in 1754 by Romulus and Remus.

If $2+2 = 5$, then you are the pope.
(a conditional fact-based declaration).

Example 1.1.4: A statement cannot be true or false unless it is declarative. This excludes commands and questions.

Go directly to jail.

What time is it?

Example 1.1.5: Declarations about semantic tokens of non-constant value are NOT propositions.

$x + 2 = 5$.

TRUTH TABLES

DEF: The ***boolean domain*** is the set $\{T, F\}$. Either of its elements is called a ***boolean value***. An n-tuple $(p_1, \ldots, p_n)$ of boolean values is called a ***boolean n-tuple***.

DEF: An n-operand ***truth table*** is a table that assigns a boolean value to the set of all boolean n-tuples.

DEF: A ***propositional operator*** is a rule defined by a truth table.

DEF: An operator is ***monadic*** if it has only one argument. It is ***dyadic*** if it has two arguments.

DEF: The following truth table defines the monadic propositional operator called ***negation***:

In other words, the negation of a proposition has the opposite truth value from the proposition itself.

Example 1.1.6: The negation operator can be used to model the following constructions:

It is **not** sunny.

$2 + 2 \neq 5.$

DEF: The following truth table defines the dyadic propositional operator called ***conjunction***:

Example 1.1.7: The word "and" is modeled by conjunction:

It is sunny **and** I am going to the beach.

Proposition 1.1.1. *The total number of propositional operators on n arguments is 2^{2^n}.*

Proof: The number of boolean n-tuples is 2^n. Thus, the number of rows in a truth table for a propositional operator on n arguments is 2^n. The truth value (in the right hand column) of each boolean n-tuple has two possibilities. $\diamondsuit$

Corollary 1.1.2. *There are* $4 = 2^2$ *monadic propositional operators.*
Corollary 1.1.3. *There are* $16 = 2^{2^2}$ *dyadic propositional operators.*

Example 1.1.8: There are only four monadic propositional operators: Identity, Negation, Constant-True, and Constant-False.

DEF: The following truth table defines the dyadic propositional operator called ***disjunction***:

The disjunction is true if either (or both) of its component clauses is true.

Example 1.1.9: Disjunction models "or":

It is rainy **or** Sweetums goes to the beach.

This sentence is TRUE if Sweetums goes to the beach in the rain. It is false only if Sweetums stays home on a sunny day.

DEF: ***Exclusive or*** is true if one and only one of its component clauses is true:

Exclusive or is GOOD for modeling binary addition. It is BAD for modeling "or" in English.

CONDITIONAL OPERATOR

DEF: This truth table defines the dyadic operator called the ***conditional*** a.k.a ***"implies"***:

In the form $p \rightarrow q$, the proposition to the left of the conditional operator (in this case, p) is called the ***antecedent***, and the proposition to the right (in this case, q) is called the ***consequent***.

The conditional operator is best understood as a CONTRACT.

Example 1.1.10: The assertion "If the Yankees win the World Series, then they give Lou Gehrig a \$1,000 bonus" is of the form $p \rightarrow q$:

p: Yankees win World Series

q: Yankees give Gehrig a \$1,000 bonus.

What circumstance would allow Gehrig to win a breach-of-contract suit against the Yankees?

Example 1.1.11: Conditional propositions.

If $2+2=4$, then Albany is the capital of NY.

If $2+2=4$, then Peapack is the capital of NJ.

If $2+2=5$, then there is a state with only one neighbor.

If $2+2=5$, then you* are the pope.

DISAMBIGUATION: ***Premises*** and ***conclusions*** are parts of logical arguments. We disambiguate them from "antecedent" and "consequent". "Hypothesis" is already an overloaded term.

DEF: The following truth table defined the dyadic operator called the ***biconditional***:

* N.B. Technically, "you" is a variable.

COMPOUND PROPOSITIONAL FORMS

DEF: A ***propositional variable*** is a variable such as p, q, r (possibly subscripted, e.g. p_j) over the boolean domain.

DEF: An ***atomic propositional form*** is either a boolean constant or a propositional variable.

DEF: A ***compound propositional form*** is derived from atomic propositional forms by application of propositional operators. Monadic operators are evaluated first, and otherwise, precedence is indicated by parentheses.

DISAMBIGUATION: A "proposition" is an instance of a propositional form. Careful terminological distinction is temporary.

Example 1.1.12: Some compound propositional forms on two variables: $\neg p$, $p \vee q$, $p \wedge q$, $p \oplus q$, $p \to q$, $p \leftrightarrow q$, $(p \vee \neg q) \to q$.

Example 1.1.13: If you don't repay the Friendly Loan Company, then you will get a call from the Unfriendly Collection Agency.

$$\neg p \to q.$$

Any compound propositional form can be evaluated by a truth table

Example 1.1.14: $(p \vee \neg q) \rightarrow q$

Example 1.1.15: From Quiz 1 in Fall 1994: Analyze $(p \wedge \neg(r \rightarrow \neg q))$ with a truth table.

SOLUTION

Example 1.1.16: §1.1 Exer 60: solve a crime.
Alice: Carlos did it. Carlos: Diana did it.
Diana: Carlos is lying. John: I didn't do it.

Oracle: Only one of them is telling the truth.
Problem: Who did it?

METHOD 1: modified truth table. Find the row in which only one statement is true.

METHOD 2: sequential analysis.
(1) If Alice is telling the truth, then so is John. Thus, Alice is lying, which implies that Carlos did not do it.
(2) Similarly, if Carlos is telling the truth, then so is John. Thus, Carlos is lying, which implies that Diana did not do it.
(3) By (2), Diana must be telling the truth.
(4) By (3), John is lying. Thus, John did it.