HW 3.3

10. The base case is clear, since $1 \cdot 1! = 2! - 1$. Assuming the inductive hypothesis, we then have

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k+1) \cdot (k+1)! &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\ &= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1, \end{aligned}$$

as desired.

14. The base case is $n = 2$, and indeed $2! < 2^2$. Assume the inductive hypothesis. Then $(k+1)! = (k+1)k! < (k+1)k^k < (k+1)(k+1)^k = (k+1)^{k+1}$.

16. The base case reduces to $6 = 6$. Assuming the inductive hypothesis we have

$$\begin{aligned} &1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\ &\quad = \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \\ &\quad = (k+1)(k+2)(k+3)\left(\frac{k}{4} + 1\right) \\ &\quad = \frac{(k+1)(k+2)(k+3)(k+4)}{4}. \end{aligned}$$

20. The statement is true for the base case, $n = 0$, since $3 \mid 0$. Suppose that $3 \mid (k^3 + 2k)$. We must show that $3 \mid ((k+1)^3 + 2(k+1))$. If we expand the expression in question, we obtain $k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$. By the inductive hypothesis, 3 divides $k^3 + 2k$, and certainly 3 divides $3(k^2 + k + 1)$, so 3 divides their sum, and we are done.