10. The base case is clear, since $1 \cdot 1! = 2! - 1$. Assuming the inductive hypothesis, we then have
\[
1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! + (k + 1) 
\cdot (k + 1)! = (k + 1)! - 1 + (k + 1) \cdot (k + 1)!
\]
\[
= (k + 1)! (1 + k + 1) - 1 = (k + 2)! - 1,
\]
as desired.

14. The base case is $n = 2$, and indeed $2! < 2^2$. Assume the inductive hypothesis. Then $(k + 1)! = (k + 1)k! < (k + 1)k^2 < (k + 1)(k + 1)^k = (k + 1)^{k+1}.$

16. The base case reduces to $6 = 6$. Assuming the inductive hypothesis we have
\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)
\]
\[
= \frac{k(k + 1)(k + 2)(k + 3)}{4} + (k + 1)(k + 2)(k + 3)
\]
\[
= (k + 1)(k + 2)(k + 3) \left( \frac{k}{4} + 1 \right)
\]
\[
= \frac{(k + 1)(k + 2)(k + 3)(k + 4)}{4}.
\]

20. The statement is true for the base case, $n = 0$, since $3 \mid 0$. Suppose that $3 \mid (k^3 + 2k)$. We must show that $3 \mid ((k + 1)^3 + 2(k + 1))$. If we expand the expression in question, we obtain $k^3 + 3k^2 + 3k + 1 + 2k + 2 = (k^3 + 2k) + 3(k^2 + k + 1)$. By the inductive hypothesis, $3$ divides $k^3 + 2k$, and certainly $3$ divides $3(k^2 + k + 1)$, so $3$ divides their sum, and we are done.